Damping Synthesis Using Complex Substructure Modes and a Hermitian System Representation

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Modal synthesis techniques have long been used to evaluate the natural frequencies and mode shapes of systems for which modal characteristics of the various components have been determined either experimentally or numerically. However, little attention has been given in the prediction of damping levels of the total structure from damping information obtained experimentally, usually in the form of modal damping ratios and complex or real mode shapes. The purpose of this paper is to present such a method, to demonstrate its use on a simple example, and to discuss two numerical aspects related to its numerical implementation: Hermitian matrices and use of transformations of the displacements only, rather than transformations twice the size incorporating velocities and displacements. Only the method of fixed constraint modes is presented herein, and damping at the joints of the substructures is not considered.

Introduction

IT is generally agreed that damping prediction, based on modal synthesis techniques utilizing only modal damping information, is inappropriate. For synthesis with real normal modes of the structural components, the off-diagonal elements of the transformed damping matrix should be used to obtain better damping estimates.[1]

Although the use of multiple shakers in sinusoidal testing is useful in determining real normal modes,[2] modern modal analysis techniques may be used to extract complex modes and the corresponding frequency and damping information from single- or multiple-point excitation techniques, either in random or sinusoidal testing.[3]

A previous Note presented a damping synthesis procedure for which both displacement and momentum quantities were included in the global and modal transformations.[4] The technique was applied to the case of free interface modes. This was later modified to a state-space representation incorporating velocities, rather than momentum quantities.[5] More recently, a consistent state-space formulation, based on a variational principle, has been proposed,[6] as well as the use of the adjoint system at the substructure level[7] both for unsymmetic systems.

The procedure developed herein considers only displacement transformation, is valid for symmetric substructure systems, and results in Hermitian global system matrices. Damping at the connections is not considered and the connections are assumed to be rigid. Fixed constraint modes are considered, but the concepts are also valid for free interface modes. The method is an extension of earlier works using fixed interface modes.[8] Methods described in Refs. 5, 6, and 8 use the transpose of the matrix of complex modes in obtaining the dynamic equations of the coupled structure, whereas the method proposed herein uses the complex conjugate transpose of the model matrix. Thus, the eigenvalues of the coupled structural system occur in complex conjugate pairs.[9]

Real Normal Modes

The structural dynamic equations for one substructure with viscous damping are partitioned in the following form:

$$\begin{bmatrix} m_{bb} & m_{bi} \\ m_{ib} & m_{ii} \end{bmatrix} \begin{Bmatrix} \ddot{x}_b \\ \ddot{x}_i \end{Bmatrix} + \begin{bmatrix} c_{bb} & c_{bi} \\ c_{ib} & c_{ii} \end{bmatrix} \begin{Bmatrix} \dot{x}_b \\ \dot{x}_i \end{Bmatrix} + \begin{bmatrix} k_{bb} & k_{bi} \\ k_{ib} & k_{ii} \end{bmatrix} \begin{Bmatrix} x_b \\ x_i \end{Bmatrix} = \begin{Bmatrix} f_b \\ f_i \end{Bmatrix} \tag{1}$$

in which the "b" index relates to boundary degrees of freedom that are in common with other substructures and "i" refers to internal degrees of freedom.

For an undamped structure, the static constraint modes and fixed interface normal dynamic modes are utilized in a local substructure transformation[10]

$$\{x\} = [\alpha]\{p\} \tag{2}$$

in which (the prime represents the transpose)

$$[\alpha] = \begin{bmatrix} [I] & [0] \\ [\phi_c] & [\phi_n] \end{bmatrix} \tag{3}$$

$$\{p\}' = \{\{p_b\}' \ \ \{p_n\}'\} \tag{4}$$

$$\{x\}' = \{\{x_b\}' \ \ \{x_i\}'\} \tag{5}$$

p_n are generalized coordinates for the fixed constraint normal modes and p_b $(=x_b)$ are boundary degrees of freedom. $[\phi_c]$ are the static response modes for unit displacements of the boundary degrees of freedom with zero internal loads $(\{f_i\}=0)$

$$[\phi_c] = -[k_{ii}]^{-1}[k_{ib}] \tag{6}$$

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$[k_{ii}]^{-1}$ is the inverse of $[k_{ii}]$. $[\phi_n]$ is a matrix of real normal modes for the dynamic behavior of the undamped substructure having all boundary degrees of freedom fixed

$$[\phi_n]'[m_{ii}][\phi_n]=[I] \tag{7}$$

$$[\phi_n]'[k_{ii}][\phi_n]=[\Omega^2] \tag{8}$$

with the prime representing the transpose. $[I]$ is the identity matrix and $[\Omega^2]$ is the diagonal matrix of the squares of undamped natural frequencies.

Normally a limited number of these dynamic modes are incorporated in the analysis, depending on the frequency range of interest. The kinetic, T, and mechanical energy U, are then approximated by

$$T=\tfrac{1}{2}\{\dot{p}\}'[\bar{m}]\{\dot{p}\} \tag{9}$$

$$U=\tfrac{1}{2}\{p\}'[\bar{k}]\{p\} \tag{10}$$

in which

$$[\bar{m}]=[\alpha]'[m][\alpha]=\left[\begin{array}{c|c}[\bar{m}_{bb}] & [\bar{m}_{bn}]\\ \hline [\bar{m}_{nb}] & [I]\end{array}\right] \tag{11}$$

$$[\bar{k}]=[\alpha]'[k][\alpha]=\left[\begin{array}{c|c}[\bar{k}_{bb}] & [0]\\ \hline [0] & [\Omega^2]\end{array}\right] \tag{12}$$

$$[\bar{k}_{bb}]=[k_{bb}]-[k_{bi}][k_{ii}]^{-1}[k_{ib}] \tag{13}$$

$$[\bar{m}_{bb}]=[m_{bb}]+[m_{bi}][\phi_c]+[\phi_c]'[m_{ib}] + [\phi_c]'[m_{ii}][\phi_c] \tag{14}$$

$$[\bar{m}_{bn}]=[\bar{m}_{nb}]'=[m_{bi}][\phi_n]+[\phi_c]'[m_{ii}][\phi_n] \tag{15}$$

In situations where the transformation is also applied to the damping matrix, the free-vibration equations of motion may be written as functions of $\{p\}$ rather than $\{x_i\}$ of Eq. (1)

$$[\bar{m}]\{\ddot{p}\}+[\bar{c}]\{\dot{p}\}+[\bar{k}]\{p\}=\{0\} \tag{16}$$

in which $[\bar{c}]$ is, in general, a full matrix

$$[\bar{c}]=[\alpha]'[c][\alpha]=\left[\begin{array}{c|c}\bar{c}_{bb} & \bar{c}_{bn}\\ \hline \bar{c}_{nb} & \bar{c}_{nn}\end{array}\right] \tag{17}$$

For certain forms of the damping, the matrix of normal modes, $[\phi_n]$, also diagonalizes the damping matrix $[c_{ii}]$[11]

$$[\bar{c}_{nn}]=[\phi_n]'[c_{ii}][\phi_n]=2[\zeta][\Omega] \tag{18}$$

in which $[\zeta]$ are the modal damping ratios. The other elements of $[\bar{c}]$ are of the same form as those of $[\bar{m}]$, except that damping matrices rather than mass matrices are used in Eqs. (14) and (15).

Damped Complex Modes

For arbitrary viscous damping, the real transformation $[\phi_n]$ does not, in general, diagonalize the damping matrix as in Eq. (18). The $[\alpha]$ transformation, Eq. (3), may be modified to the following:

$$[\alpha]=\left[\begin{array}{c|c}[I] & [0]\\ \hline [\phi_c] & [\phi_d]\end{array}\right] \tag{19}$$

in which the only difference is the use of damped mode shapes $[\phi_d]$ rather than the real normal modes $[\phi_n]$. These are, in general complex, involving phase relationships, and are obtained from the first-order eigenvalues problem

$$[[a]-\lambda[b]]\{v\}=\{0\} \tag{20}$$

in which both velocities and displacements are considered at the substructure level

$$\{v\}=\left\{\begin{array}{c}x\\ \hline \dot{x}\end{array}\right\} \tag{21}$$

$$[a]=\left[\begin{array}{c|c}[k] & [0]\\ \hline [0] & -[m]\end{array}\right] \tag{22}$$

$$[b]=\left[\begin{array}{c|c}-[c] & -[m]\\ \hline -[m] & [0]\end{array}\right] \tag{23}$$

For underdamped systems, the eigenvalues and eigenvectors occur in complex conjugate pairs and the matrix of eigenvalues $[\Lambda]$ are eigenvectors $[V]$ may be partitioned as follows[11]:

$$[\Lambda]=\left[\begin{array}{c|c}[\lambda] & [0]\\ \hline [0] & [\lambda^*]\end{array}\right] \tag{24}$$

$$[V]=\left[\begin{array}{c|c}[\phi_d] & [\phi_d^*]\\ \hline [\phi_d][\lambda] & [\phi_d^*][\lambda^*]\end{array}\right] \tag{25}$$

Modal Synthesis

In the substructure transformation, Eq. (2), the boundary degrees of freedom were kept and the internal degrees of freedom were transformed to generalized coordinates corresponding to the fixed mode shapes This transformation is repeated for each substructure. Certain boundary degrees of freedom are common to two or more substructures. A global transformation is then used to incorporate this fact. Each of the generalized dynamic coordinates of each substructure are, however, left intact, $\{p_d\}_\ell=\{q_d\}_\ell$, $\ell=1,2,\ldots,N$, for N the number of substructures and the subscript d for damped complex modes

$$\{p\}'=\{\{p_b\}_1'\{p_d\}_1'\{p_b\}_2'\{p_d\}_2'\ldots\{p_b\}_N'\{p_d\}_N'\} \tag{26}$$

$$\{q\}'=\{\{q_d\}_1'\{q_d\}_2'\ldots\{q_d\}_N'\{q_b\}'\} \tag{27}$$

The global transformation $[\beta]$ is defined as

$$\{p\}=[\beta]\{q\} \tag{28}$$

The global matrices are then given by

$$[M]=\sum_{r=1}^{N}[\beta]_r'[\bar{m}]_r[\beta]_r \tag{29}$$

$$[K]=\sum_{r=1}^{N}[\beta]_r'[\bar{k}]_r[\beta]_r \tag{30}$$

$$[C]=\sum_{r=1}^{N}[\beta]_r'[\bar{c}]_r[\beta]_r \tag{31}$$

in which $[\beta]_r$ is the portion of $[\beta]$ in Eq. (28) corresponding to the rth substructure

$$[\beta]_r = \left[\begin{array}{c|c|c|c|c|c} [0] & \ldots & [0] & \ldots & [0] & [\beta_b]_r \\ \hline [0] & \ldots & [I]_r & \ldots & [0] & [0] \end{array}\right] \tag{32}$$

with $[\beta_b]_r$ representing the compatibility conditions for the degrees of freedom at the boundaries of the substructure "r."

Global Dynamic Equations

The global mass, damping, and stiffness matrices utilizing the global transformation $[\beta]$, Eq. (28), and the damped modes of the substructures, $[\alpha]_r$, Eq. (19), for $r=1,2,\ldots,N$, are formulated to be Hermitian, i.e., equal to their complex conjugate transpose,

$$[M] = \sum_{r=1}^{N} [\beta]_r'[\alpha]_r^+[m]_r[\alpha]_r[\beta]_r \tag{33}$$

$$[K] = \sum_{r=1}^{N} [\beta]_r'[\alpha]_r^+[k]_r[\alpha]_r[\beta]_r \tag{34}$$

$$[C] = \sum_{r=1}^{N} [\beta]_r'[\alpha]_r^+[c]_r[\alpha]_r[\beta]_r \tag{35}$$

in which $[\alpha]_r^+$ represents the complex conjugate transpose of the local transformation of the rth substructure. The matrices are of the form

$$[M] = \begin{bmatrix} [M_{dd}]_1 & & & & [M_{db}]_1 \\ & [M_{dd}]_2 & & [0] & [M_{db}]_2 \\ & & \ddots & & \vdots \\ & [0] & & & \\ & & & [M_{dd}]_N & [M_{db}]_N \\ [M_{bd}]_1 & [M_{bd}]_2 & \ldots & [M_{db}]_N & [M_{bb}] \end{bmatrix} \tag{36}$$

$$[K] = \begin{bmatrix} [K_{dd}]_1 & & & \\ & [K_{dd}]_2 & [0] & \\ & & [K_{dd}]_N & \\ & [0] & & \\ & & & [K_{bb}] \end{bmatrix} \tag{37}$$

$$[C] = \begin{bmatrix} [C_{dd}]_1 & & & & [C_{db}]_1 \\ & [C_{dd}]_2 & & [0] & [C_{db}]_2 \\ & & \ddots & & \vdots \\ & [0] & & & \\ & & & [C_{dd}]_N & [C_{db}]_N \\ [C_{bd}]_1 & [C_{bd}]_2 & \ldots & [C_{db}]_N & [C_{bb}] \end{bmatrix} \tag{38}$$

in which the subscript d refers to the damped complex modes of each substructure and

$$[M_{dd}]_r = [\phi_d]_r^+[m_{ii}]_r[\phi_d]_r \tag{39}$$

$$[M_{bb}] = \sum_{r=1}^{N} [\beta_b]_r'[\bar{m}_{bb}]_r[\beta_b]_r \tag{40}$$

$$[M_{bd}]_r = [M_{db}]_r^+ = [\beta_b]_r'[\bar{m}_{bd}]_r \tag{41}$$

$$[K_{dd}]_r = [\phi_d]_r^+[k_{ii}]_r[\phi_d]_r \tag{42}$$

$$[K_{bb}] = \sum_{r=1}^{N} [\beta_b]_r'[\bar{k}_{bb}]_r[\beta_b]_r \tag{43}$$

$$[C_{dd}]_r = [\phi_d]_r^+[c_{ii}]_r[\phi_d]_r \tag{44}$$

$$[C_{bb}] = \sum_{r=1}^{N} [\beta_b]'[\bar{c}_{bb}]_r[\beta_b]_r \tag{45}$$

$$[C_{bd}]_r = [C_{db}]_r^+ = [\beta_b]_r'[\bar{c}_{bd}]_r \tag{46}$$

in which the (¯) matrices for each substructure are given in Eqs. (13-18), except that $[\phi_d]$ and $[\phi_d]^+$ are used instead of $[\phi_n]$ and $[\phi_n]'$. The global transformation $[\beta]$ and the static response modes $[\phi_c]$ are real, but the $[\phi_d]$ are complex quantities.

The global dynamic equation is then expressed in first-order form similar to Eqs. (20-23)

$$[A]\{s\} - [B]\{\dot{s}\} = 0 \tag{47}$$

in which

$$[A] = \left[\begin{array}{c|c} [K] & [0] \\ \hline [0] & -[M] \end{array}\right] \tag{48}$$

$$[B] = \left[\begin{array}{c|c} -[C] & -[M] \\ \hline -[M] & -[0] \end{array}\right] \tag{49}$$

$$\{s\} = \left\{\begin{array}{c} q \\ \hline \dot{q} \end{array}\right\} \tag{50}$$

Both $[A]$ and $[B]$ are Hermitian matrices, i.e., their complex conjugate transpose is equal to the matrices and thus the eigenvalues of the complete structure are real or occur in complex conjugate pairs. This is easily shown by considering

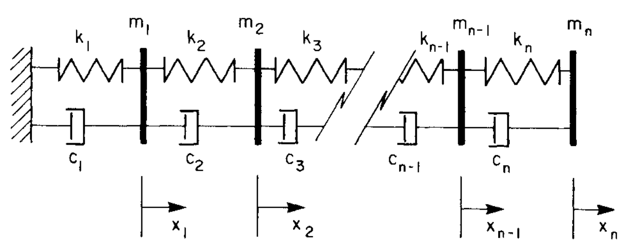

Fig. 1 Structure ($n=20$).

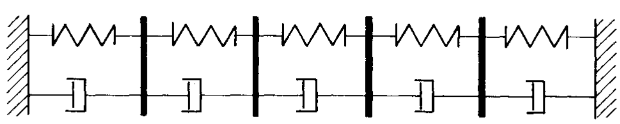

Fig. 2 Substructures 1-3.

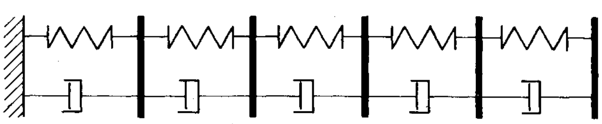

Fig. 3 Substructure 4.

the complex conjugate transpose of the determinant obtained from an eigenvalue consideration of Eq. (47) (see the Appendix).

$$|[[A]-\lambda[B]]|^{+}=|[A]^{+}-\lambda^{*}[B]^{+}|$$
$$=|[A]-\lambda^{*}[B]|=0 \tag{51}$$

Example

A simply coupled structure with 20 degrees of freedom was used to compare the method outlined herein with other existing techniques by numerical simulation. The structure was composed of four substructures, each having five degrees of freedom, as shown in Figs. 1-3. The stiffness and mass are arbitrarily set equal to 1 and the damping values are given in Table 1. The mass, stiffness, and damping matrices of the total structure are of the following form:

$$[m]=m\begin{bmatrix}1 & & & & \\ & 1 & & [0] & \\ & & \cdot & & \\ & [0] & & \cdot & \\ & & & & \cdot \\ & & & & & 1\end{bmatrix} \tag{52}$$

$$[k]=k\begin{bmatrix}2 & -1 & & & & \\ -1 & 2 & -1 & [0] & & \\ & -1 & 2 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & [0] & -1 & 2 & -1 & \\ & & & -1 & 1 & \end{bmatrix} \tag{53}$$

$$[c]=\begin{bmatrix}c_1+c_2 & -c_2 & & & & \\ & & & [0] & & \\ -c_2 & c_2+c_3 & -c_3 & & & \\ & -c_3 & \cdot & \cdot\ \cdot & & \\ [0] & & & c_{19}+c_{20} & -c_{20} & \\ & & & -c_{20} & c_{20} & \end{bmatrix} \tag{54}$$

Four different synthesis techniques were used, all with fixed interface modes. For all of the four methods described below, only two dynamic modes were used in each of the substructures together with the static constraint modes.

Method 1. Real normal substructure modes neglecting the off-diagonal elements of the reduced damping matrix $[\bar{c}_{nn}]$, Eq. (17).

Method 2. Real normal substructure modes including the off-diagonal elements of the reduced damping matrix.

Method 3. Damped complex substructure modes for which both velocities and displacements are transformed.

Method 4. Dampled complex substructure modes for which only the displacements are transformed. This is the method proposed herein.

Method 3 is an adaptation of the method described in Ref. 8 applied to the state-space formulation in the case of fixed interface modes. The complex conjugate transpose of $[\alpha]$ is used, however, rather than the transpose as proposed initially. $[\alpha]$ relates the x and p coordinates, Eq. (2),[8]

Table 1 Damping coefficients

Degrees of freedom	Viscous daming, c	Degrees of freedom	Viscous damping, c
1	0.10	11	0.09
2	0.50	12	0.50
3	0.08	13	0.07
4	0.60	14	0.11
5	0.09	15	0.50
6	0.11	16	0.09
7	0.50	17	0.07
8	0.12	18	0.40
9	0.09	19	0.11
10	0.07	20	0.50

$$\begin{Bmatrix}x_b\\ \dot{x}_b\\ x_i\\ \dot{x}_i\end{Bmatrix}=\begin{bmatrix}[I] & [0] & [0]\\ [0] & [I] & [0]\\ [\phi_c] & [0] & [\phi_d]\\ [0] & [\phi_c] & [\phi_d][\lambda]\end{bmatrix}\begin{Bmatrix}p_b\\ \dot{p}_b\\ p_d\end{Bmatrix} \tag{55}$$

This technique requires more numerical manipulation and computer storage than the other methods, which consider the transformation of displacement degrees of freedom only. Similarly, in method 4, a Hermitian system is obtained by using the Hermitian of $[\alpha]$ in the formulation of the global $[A]$ and $[B]$ matrices, rather than the transpose. The order of the matrices $[A]$ and $[B]$ of Eq. (47) is the same for all four methods, 22×22, since there are eight complex mode shapes and three boundary degrees of freedom. The subroutine EIGZC was used to solve eigenvalues of Eq. (47)[12] and did not consider the fact that the matrices $[A]$ and $[B]$ are Hermitian, since it solves the generalized eigenvalue problem for arbitrary complex matrices.

The velocity transformation, Eq. (3), is not needed in synthesis applications, since all of the necessary information is contained in $[\phi_d]$, the top left $n\times n$ matrix of Eq. (25). Similarly, the complex conjugate mode shapes, i.e., the lower $n\times n$ matrix of Eq. (25), is not required.

The results are compared to values obtained numerically from the whole structure. The real and imaginary portions of the complex conjugate pairs of eigenvalues are given in Tables 2 and 3, respectively, with the error of each method expressed as a percentage of the value indicated in the left column. Only the smallest five frequencies are tabulated

$$\lambda=-\sigma+i\omega \tag{56}$$

The four method give quite acceptable results for the imaginary portion of the eigenvalues, corresponding to the damped natural frequencies. The first method, however, does not synthesize the damping very well, as is seen in Table 2. The use of either normal or complex modes in the synthesis process appears to be accurate enough, at least for this example. Accuracy of the mode shapes is also good for all of the methods.

The computer times of the central processing unit of an IBM 370/175 to obtain these results are given in Table 4. Method 3 requires over twice the computer time of method 4. Method 2, which keeps the off-diagonal damping terms, is slightly slower than method 1, and the technique proposed here, using complex modes, takes about one-third more time than the methods that use real normal modes.

Table 2 Real portion of eigenvalues (σ, damping)

Reference, σ	Modal synthesis results, relative error, % Method 1	Method 2	Method 3	Method 4
0.000689	0.73	0.44	0.29	0.29
0.006108	0.33	−0.54	−1.26	−0.56
0.014764	−2.38	−0.14	−1.93	−0.83
0.032430	−10.27	1.26	−0.29	0.06
0.044951	3.81	2.26	−0.05	0.85

Table 3 Imaginary portion of eigenvalues (ω, frequency)

Reference, ω	Modal synthesis results, relative error, % Method 1	Method 2	Method 3	Method 4
0.012193	0	0	0	0
0.036530	−0.06	−0.06	−0.06	−0.03
0.060701	−0.12	−0.09	−0.05	0.04
0.084646	−0.33	−0.34	−0.04	0.05
0.108321	−0.53	−0.34	0.05	0.11

Table 4 Computer time

Method	Exact	1	2	3	4
Time	28.5	6.8	6.9	21.4	9.2

Conclusion

The proposed numerical procedure for the synthesis of damping and frequency as well as mode shapes is shown to be as good as existing methods. Only displacements are transformed, and velocities need not be transformed, as in a state formulation. It uses complex mode shapes rather than real normal modes for the substructures. The global stiffness, mass, and damping matrices are obtained to yield a Hermitian system by use of the complex conjugate transpose of the local transformation matrix, rather than its transpose. This then yields complex conjugate pairs of eigenvalues in underdamped dynamic systems.

Appendix: Proof of Complex Conjugate Pairs of Eigenvalues[9]

For both $[A]$ and $[B]$ Hermitian, i.e.,

$$[A]^+ = [A] \tag{A1}$$

$$[B]^+ = [B] \tag{A2}$$

$[B]$ then may be diagonalized by a unitary matrix $[U]$

$$[B] = [U][\Lambda]_B[U]^+ \tag{A3}$$

in which $[\Lambda]_B$ are the real eigenvalues of $[B]$ and $[U]$ is a unitary matrix

$$[U]^{-1} = [U]^+ \tag{A4}$$

The eigenvalue problem

$$[A]\{x\} = \Lambda[B]\{x\} \tag{A5}$$

may then be reformulated as

$$[D][C][D]\{y\} = \Lambda\{y\} \tag{A6}$$

in which $[C]$ is Hermitian

$$[C] = [U]^+[A][U] \tag{A7}$$

$$[D] = [\Lambda_B]^{-1/2} \tag{A8}$$

$$\{x\} = [U][D]\{y\} \tag{A9}$$

and $[\Lambda_B]^{-1/2}$ is the diagonal matrix of 1 over the square root of the real (positive or negative) values of $[B]$. The sum of the eigenvalues Λ is real, since the pre- and postmultiplication of a Hermitian matrix by the diagonal matrix D will yield a matrix with a real trace. The product of the eigenvalues Λ is also real, since the determinant of $[D]$ $[C]$ $[D]$ is real due to the fact that the eigenvalues of $[C]$ are real and the square of the diagonal elements of $[D]$ is also real. Since the sum and product of the eigenvalues Λ are both real, the eigenvalues must be real or occur in complex conjugate pairs.

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